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Expo. Math. 30 (2012) 124–153

EXPOSITIONES MATHEMATICAE

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Harmonic analysis of weighted L^p-algebras

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Received 27 January 2011

Abstract

Let G be a locally compact, compactly generated group of polynomial growth and let ω be a weight on G. Under proper assumptions on the weight ω, the Banach space $L^p(G, \omega)$ is a Banach $*$-algebra. In this paper we give examples of such weighted L^p-algebras and we study some of their harmonic analysis properties, such as symmetry, existence of functional calculus, regularity, weak Wiener property, Wiener property, and existence of minimal ideals of a given hull.
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MSC 2000: 43A15; 22D15; 22D20

Keywords: Weighted L^p-algebras; Symmetry; Functional calculus; Regularity; Weak Wiener property; Wiener property; Minimal ideals of a given hull

1. Introduction

Weights and weighted function spaces play an important role in mathematics. In essence, a weight makes it possible to study the behaviour of functions around a certain point, ignoring their oscillations at infinity, or on the contrary, to amplify the asymptotic behaviour of a function. More precisely, introducing a weight means modelling in a quantitative manner the decay of the functions to be studied. This has numerous applications in numerical mathematics and is quite often used for concrete applications (signal theory, Gabor analysis, sampling theory, etc.); see for instance [19,8,20,15].

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doi:10.1016/j.exmath.2012.01.002

On the other hand, weights appear naturally in analysis: in inequalities relating the norm of a function to the norm of its derivatives, in extension theorems, etc.; see, e.g., a survey of Kudryavtsev and Nikol'skiĭ [26]. One of the areas where weighed spaces are applied most intensively is the theory of boundary value problems for partial differential equations (see the surveys [26,17]). By the way, using the Laplace transform also means working in a weighted function space.

In representation theory, which interests us most, weights occur for instance in the following way. If G denotes a locally compact group and (T, V) is a continuous representation of G, then the maps $\omega : x \mapsto \|T(x)\|_{op}$ and $\omega : x \mapsto \max(\|T(x)\|_{op}, \|T(x^{-1})\|_{op})$ are weights, the last one being symmetric ($\omega(x^{-1}) = \omega(x),\ \forall x$). For any one of these weights, the map

$$f \mapsto T(f) := \int_G f(x)T(x)dx$$

is a representation of the weighted function algebra

$$L^1(G, \omega) := \left\{ f : G \to \mathbb{C} \mid f \text{ measurable and } \int_G |f(x)|\omega(x)dx < +\infty \right\}.$$

Sub-exponential weights like the ones introduced in 1.2.3 below, appear in the context of nilpotent Lie groups. In fact, let G be a connected, nilpotent Lie group. Let G_1 be the derived group of G, i. e. the closed subgroup generated by the elements of the form $[x, y] = x^{-1}y^{-1}xy,\ x, y \in G$. Let U be a generating neighbourhood of the identity e in G and $V = U \cap G_1$ the corresponding neighbourhood of e in G_1. Let

$$|x|_U := \inf\{n \in \mathbb{N} \mid x \in U^n\} \quad \text{for } x \neq e \text{ and } |e|_U = 1,$$

similarly for $|x|_V$. Then it is shown in [2]), that for any weight ω on G which is submultiplicative, i.e. such that $\omega(xy) \le \omega(x)\omega(y)$ for all x, y,

$$\omega|_{G_1}(x) \le e^{C|x|_V^{\frac{1}{2}}}, \quad \forall x \in G_1,$$

for some constant C. By the way, on any compactly generated locally compact group, with generating neighbourhood U, every submultiplicative weight ω is exponentially bounded, i.e. satisfies a relation of the form $\omega(x) \le e^{K|x|_U}$ for $K = \ln \sup_{x\in U} \omega(x)$.

If the weight ω is submultiplicative, then the weighted function space $L^1(G, \omega)$ is a Banach algebra for convolution, and even a Banach $*$-algebra if the weight is symmetric. The advantage of Banach $*$-algebras over just Banach spaces is clear. They have a much richer structure which may be studied via representation theory and harmonic analysis techniques. In this way, interesting problems arise. Let us just mention the question of their ideal theory, problems of generalized spectral synthesis, of symmetry of the algebra, of invertibility, and of factorization problems. All these questions make sense for the weighted algebra $L^1(G, \omega)$. Moreover, it is the harmonic analysis properties for algebras that makes the weighted algebras $L^1(G, \omega)$ interesting for some of the concrete applications mentioned in the beginning (see for instance [20]).

On the other hand, the importance of L^p-spaces of the form $L^p(G)$ or $L^p(G, \omega)$, $1 < p < +\infty$, is well known in functional analysis. It would be attractive to extend the theory of convolution algebras to the L^p-case, because L^p spaces are reflexive — not a common property among Banach algebras. Unfortunately, if G is not compact and if $p \neq 1$, $L^p(G)$ is not an algebra for convolution. For locally compact abelian groups this so-called L^p-conjecture was first proved by Żelasko [50] and simplified by Urbanik [46]. The final proof for the non-abelian case has been given by Saeki [43]. Nevertheless, for appropriate groups G and weights ω, the weighted L^p-spaces

$$L^p(G,\omega) := \Bigg\{ f : G \to \mathbb{C} \mid f \text{ measurable and } \|f\|_{p,\omega} := \left(\int_G |f(x)|^p \omega(x)^p dx\right)^{\frac{1}{p}} < +\infty \Bigg\}$$

may be algebras. A sufficient condition on the group G for the existence of weighted L^p-algebras, is that the group is σ-compact. In that case, there are even a lot of such weighted L^p-algebras. In 1.2.2 we show that any positive symmetric submultiplicative function multiplied by any L^p-algebra weight produces again an L^p-algebra weight. This makes it possible to construct L^p-algebra weights with all kinds of different growth behaviours.

In the context of weighted L^p-algebras, let us mention the works of Wermer [48], Nikol'skiĭ [39], Feichtinger [14], and recently Kuznetsova [27–30]. Most of these papers concentrate mainly on the question whether the corresponding L^p-spaces are algebras. The only well-studied case is $L^p(\mathbb{Z},\omega)$, see, e.g., a long paper of El-Fallah et al. [13]. This is mainly for the reason that in the problems of weighted approximation by polynomials, as initiated by Bernstein [3], $L^p(\mathbb{Z},\omega)$ algebras play a distinguished role [39]. But this is not the only possible application of weighted L^p-algebras. Similarly as for L^1-algebras, the weights can be used in numerical mathematics to model the decay of the functions to be used and allow numerical computations. On the other hand, weighted L^p-algebras may turn out to be important examples for people working in Banach algebra theory or operator theory. Such algebras have already been used successfully in the interpolation theory and in questions of factorization [6].

As for possible harmonic analysis properties of an arbitrary Banach $*$-algebra $\mathcal{A}$, several questions have particularly caught the interest of mathematicians:

Is the algebra symmetric, i.e. does every self-adjoint element have a real spectrum?

Is the algebra $\mathcal{A}$ regular, i.e. do the elements of the algebra separate points from closed sets in $\mathrm{Prim}_*\mathcal{A}$, the space of kernels of topologically irreducible unitary representations?

Does the algebra have the weak Wiener property, i.e. is every proper, closed, two-sided ideal annihilated by an algebraically irreducible representation?

Does the algebra have the Wiener property, i.e. does the previous property hold for topologically irreducible unitary representations?

Do minimal ideals with a given hull exist?

Details about these properties and their meaning in the abelian case will be given in the corresponding sections of this paper.

It is not possible to give a complete list of authors who studied group algebras $L^1(G)$ and their properties. For weighted group algebras $L^1(G,\omega)$, let us mention among others the following: In the abelian case, the systematic study of such properties for weighted group algebras $L^1(G,\omega)$ goes back to Beurling [4,5], Domar [10] and Vretblad [47] among others. In the non-abelian case, one may refer to Hulanicki [22,23], Pytlik [40,41], as well as to more recent studies [11,16,15]. In [15] for instance, the question of the symmetry for weighted group algebras $L^1(G, \omega)$ is completely solved for compactly generated groups with polynomial growth. Let us mention in this respect, that these abstract problems may be quite important for concrete applications. For instance, Gröchenig and Leinert [19] point out that the theory of symmetric group algebras is an important tool to solve problems about Gabor frames, motivated by signal theory.

A systematic study of harmonic analysis properties of weighted L^p-algebras $L^p(G, \omega)$ should also be of importance, as well for applications and for more abstract mathematical problems. In [29] some harmonic analysis properties like the regularity are studied in the case of abelian groups. But not much seems to have been done up to now in the non-abelian case. Hence the main purpose of this paper will be to study harmonic analysis properties in the context of non-abelian weighted L^p-algebras.

In the present paper, we work on general compactly generated groups with polynomial growth. The weight ω is supposed to satisfy $\omega^{-q} * \omega^{-q} \leq C\omega^{-q}$ for some constant $C > 0$, where $\frac{1}{p} + \frac{1}{q} = 1$. This ensures $L^p(G, \omega)$ to be an algebra [48,27]. See Section 1.1 for a short outline of proof. We also assume the weight to be submultiplicative. We start by giving examples of $L^p(G, \omega)*$-algebras for polynomially growing weights as well as for sub-exponentially growing weights. We then address the questions raised previously: We prove the symmetry of the L^p-algebra $L^p(G, \omega)$, if either G is abelian and ω satisfies the same condition as for the case of $L^1(G, \omega)$ (condition (S), [16,15]) or if ω is polynomial in the sense of Pytlik [41] (and G not necessarily abelian). The same hypothesis as in [11], i.e. the non-abelian Beurling–Domar condition (BDna), allows us to construct a functional calculus on a total subset of the algebra $L^p(G, \omega)$ and to show regularity, as well as the weak Wiener property. If the L^p-algebra is moreover symmetric, we also get the Wiener property and the existence of minimal ideals of a given hull. Let us recall that the (BDna) condition is defined as follows in [11]: Let $G = \bigcup_n U^n$, where U is a relatively compact, generating neighbourhood of e in G. We define $s(n) := \sup_{x \in U^n} \omega(x)$. Then the weight ω satisfies (BDna) if and only if

$$\sum_{n \in \mathbb{N}, n \geq e^e} \frac{\big(\ln(\ln n)\big) \ln\big(s(n)\big)}{1 + n^2} < +\infty.$$

This condition is independent of the choice of the generating neighbourhood U and is only slightly stronger than the conditions used by Domar and Beurling in the abelian case (see Section 4.2 and [11] for additional comments on this condition). These results rely on the corresponding results for L^1-algebras $L^1(G, \omega)$. The question of whether, more generally, condition (S) or the GRS-condition as defined in [16,15] implies the symmetry of the algebra $L^p(G, \omega)$, as they do for $L^1(G, \omega)$, is still an open problem (see Section 3). Finally, let us point out that although these algebras $L^p(G, \omega)$ have certain nice harmonic analysis properties, they are not amenable if $p > 1$ and G non-discrete, as they do not have bounded approximate identities [28].

Let us also mention, that we assume our weights to be submultiplicative, in order for $L^1(G, \omega)$ to be a $*$-algebra. Therefore we can rely on the known $L^1(G, \omega)$-results. But there are examples of weights which are not submultiplicative and which produce nevertheless Banach $*$-algebras $L^p(G, \omega)$ (see for instance [28]). Studying these weights and the properties of the corresponding L^p-algebras is still a challenge.

1.1. Assumptions on groups and weights

We suppose in this paper that G is a compactly generated locally compact group. This group G is a group of *polynomial growth* if there is a relatively compact generating neighbourhood of the identity U such that $|U^n| \leq Cn^Q$ for some constants C, Q, where $|U^n|$ denotes the Haar measure of U^n. It is known that Q does not depend on the choice of U. The class of such groups will be denoted by $[PG]$, and for the rest of the paper we assume that G is in $[PG]$.

If U is a relatively compact generating neighbourhood of the identity, we define

$$|x|_U := \inf\{n \mid x \in U^n\}.$$

When the choice of U is not important, we write simply $|x|$. In the case $G = \mathbb{R}$, $|x|$ may also denote the absolute value of x, and the results of this paper remain correct.

Let

$$\omega : G \to [1, +\infty[$$

be a measurable function (*weight*) such that

$$\begin{aligned}
&\omega(xy) \leq \omega(x)\omega(y), \quad \forall x, y \in G\\
&\omega(x) = \omega(x^{-1}), \quad \forall x \in G\\
&\omega^{-q} * \omega^{-q} \leq \omega^{-q},
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$. These conditions are sufficient for $L^p(G, \omega)$ to be a Banach $*$-algebra with the usual involution, defined by $f^*(x) := \overline{f(x^{-1})}$ in the unimodular case [48,27,30]. We will say that (G, ω) satisfies (LPAlg) (L^p-algebra) if these conditions are satisfied.

For the sake of completeness, let us sketch a proof of the fact that the condition $\omega^{-q} * \omega^{-q} \leq \omega^{-q}$ implies $L^p(G, \omega)$ to be a Banach algebra. A similar proof for the case of $\mathbb{R}^n$ is for instance found in [25]. For $f, g \in L^p(G, \omega)$ we have

$$\begin{aligned}
\|f * g\|_{p,\omega}^p &= \int_G \omega^p(x) \left| \int_G f(y) g(y^{-1}x) dy \right|^p dx\\
&= \int_G \omega^p(x) \left| \int_G f(y)\omega(y) g(y^{-1}x)\omega(y^{-1}x) \frac{1}{\omega(y)\omega(y^{-1}x)} dy \right|^p dx\\
&\leq \int_G \omega(x)^p \left(\left(\int_G |f(y)|^p \omega(y)^p |g(y^{-1}x)|^p \omega(y^{-1}x)^p dy \right)^{\frac{1}{p}} \right.\\
&\quad \left. \cdot \left(\int_G \frac{1}{\omega(y)^q \omega(y^{-1}x)^q} dy \right)^{\frac{1}{q}} \right)^p dx\\
&= \int_G \omega(x)^p \left(\omega^{-q} * \omega^{-q}(x) \right)^{\frac{p}{q}}\\
&\quad \left(\int_G |f(y)|^p \omega(y)^p |g(y^{-1}x)|^p \omega(y^{-1}x)^p dy \right) dx\\
&\leq \|f\|_{p,\omega}^p \|g\|_{p,\omega}^p
\end{aligned}$$

as $\omega^{-q} * \omega^{-q} \leq \omega^{-q}$. The inequality in line 3 of the computation is due to Hölder's inequality. This completes the proof.

It is often easier to check that ω satisfies conditions (LPAlg) with some constants C_1, C_2: $\omega(xy) \leq C_1\omega(x)\omega(y)$ and $\omega^{-q} * \omega^{-q} \leq C_2\omega^{-q}$. But a renormalizing $\omega_1 = C\omega$ with $C = \max(C_1, C_2^{1/q})$ gives an equivalent weight satisfying (LPAlg).

Since every group in $[PG]$ is amenable, it follows from (LPAlg), by [30, Theorem 3.2], that $\omega^{-q} \in L^1(G)$. This implies that $L^p(G, \omega) \subset L^1(G)$, by Hölder's inequality. We may assume, without loss of generality, that the weight ω is continuous [14]. This will be assumed for the rest of the paper, except for some examples which depend on the discontinuous function $|x| = |x|_U$.

1.2. Examples of weights

1.2.1. Polynomial weights

On every group of polynomial growth, the weight $\omega(x) = (1 + |x|)^D$ satisfies (LPAlg) for D sufficiently large [14], so $L^p(G, \omega)$ is an algebra.

1.2.2. Products of weights

Let u be a positive submultiplicative function on G such that $u(x) = u(x^{-1})$, and let w_1 be a weight satisfying (LPAlg). Then $w(x) = u(x)w_1(x)$ also satisfies (LPAlg). In particular, any such submultiplicative function u, multiplied by $(1+|x|)^D$ for D sufficiently large, is a (LPAlg)-weight. To prove this, we need to check only the last condition:

$$\begin{aligned}(w^{-q} * w^{-q})(x) &= \int_G w^{-q}(y)w^{-q}(y^{-1}x)dy \\ &= \int_G u^{-q}(y)u^{-q}(y^{-1}x)w_1^{-q}(y)w_1^{-q}(y^{-1}x)dy.\end{aligned}$$

From $u(x) \leq u(y)u(y^{-1}x)$ we have $u(x)^{-q} \geq u(y)^{-q}u(y^{-1}x)^{-q}$, so the integral above is bounded by

$$(w^{-q} * w^{-q})(x) \leq u^{-q}(x) \int_G w_1^{-q}(y)w_1^{-q}(y^{-1}x)dy \leq u^{-q}(x)w_1^{-q}(x) = w^{-q}(x).$$

1.2.3. Non-polynomial weight

By the reasoning of Section 1.2.2, $\omega(x) = e^{|x|^\gamma}(1 + |x|)^D$ with $0 < \gamma \leq 1$ is an L^p-algebra weight on any group in $[PG]$ for all D sufficiently large.

Moreover, it can be shown that the weight $\omega(x) = e^{|x|^\gamma}$ itself with $0 < \gamma < 1$ satisfies (LPAlg) for all $p > 1$. For $G = \mathbb{R}$ this example is contained in the very first paper of Wermer [48] on L^p-algebras.

Let $G = \cup U^n$, where $U = U^{-1}$ and $|U^n| \leq Cn^Q$. Denote $U_n = U^n \setminus U^{n-1}$ and assume that G is non-compact. We define

$$\omega_n := \omega|_{U_n} = e^{n^\gamma},$$

$0 < \gamma < 1$, and show that $L^p(G, \omega)$ is an algebra for every $p > 1$ (the case $p = 1$ is known). For this, we check the sufficient condition $\omega^{-q} * \omega^{-q} \leq C'\omega^{-q}$.

Set $u := \omega^{-q}$, then

$$u_n := u|_{U_n} = e^{-qn^\gamma}.$$

Take $x \in U_m$ and compute $\frac{u*u(x)}{u(x)}$:

$$\frac{u*u(x)}{u(x)} = \frac{1}{u_m}\int_G u(y)u(y^{-1}x)dy = \sum_n \frac{u_n}{u_m}\int_{U_n} u(y^{-1}x)dy.$$

If $y \in U_n = U_n^{-1}$, and $y^{-1}x \in U_k$, then $\max(n-m, m-n) \le k \le n+m$. Set $U_{nk}^x := U_k \cap (U_n x)$. Then $U_n x = \cup_k U_{nk}^x$, $U_k = \cup_n U_{nk}^x$. In particular, $|U_{nk}^x| \le \min(|U_n|, |U_k|)$. We can rewrite the sum as

$$\frac{u*u(x)}{u(x)} = \sum_{n,k}\frac{u_n}{u_m} u_k |U_{nk}^x|.$$

Note that u_n decreases and $C_0 = \int_G u < \infty$. Split now the sum into four parts:

(1) $n \ge m$. Then $u_n \le u_m$, and

$$(\text{sum1}) \le \sum_{n,k}\frac{u_m}{u_m} u_k |U_{nk}^x| \le \sum_k u_k |U_k| = \int_G u = C_0.$$

(2) Similarly, if $n < m$ but $k \ge m$ then $u_k \le u_m$ and

$$(\text{sum2}) \le \sum_{n,k}\frac{u_m}{u_m} u_n |U_{nk}^x| \le \sum_n u_n |U_n| = \int_G u = C_0.$$

(3) $m/2 < n < m$, $m/2 < k < m$. Then $\max(u_n, u_k) \le u_{[m/2]+1} \le \exp(-qm^\gamma/2^\gamma)$;

$$\begin{aligned}(\text{sum3}) &\le \sum_{n=[m/2]+1}^{m}\ \sum_{k=[m/2]+1}^{m} e^{q(m^\gamma - 2m^\gamma 2^{-\gamma})}|U_{n,k}^x| \\ &\le \sum_{n=1}^{m} e^{qm^\gamma(1-2^{1-\gamma})}|U_n| \le e^{qm^\gamma(1-2^{1-\gamma})} C m^Q \cdot m.\end{aligned}$$

Since $2^{1-\gamma} > 1$, the coefficient in the exponent is negative, so this expression tends to zero as $m \to \infty$. Thus, this is bounded by a constant C_1.

(4) The only complicated case is $n \le m/2$, $m-n \le k < m$, and the symmetric case $k \le m/2$, $m-k \le n < m$ which reduces to the first one by exchanging k and n. Here $u_k \le u_{m-n}$, so

$$\begin{aligned}(\text{sum4}) &\le \sum_{n=0}^{[m/2]}\frac{u_n u_{m-n}}{u_m}\sum_{k=m-n}^{m}|U_{nk}^x| \le \sum_{n=0}^{[m/2]}\frac{u_n u_{m-n}}{u_m}|U_n| \\ &\le C\sum_{n=0}^{[m/2]} e^{q(m^\gamma - n^\gamma - (m-n)^\gamma)} n^Q.\end{aligned}$$

Denote

$$f(x) = \int_0^{x/2} t^Q e^{q(x^\gamma - t^\gamma - (x-t)^\gamma)} dt;$$

clearly our sum is bounded for all m if and only if f is bounded on $\mathbb{R}_+$.

By changing the variable to $s = t/x$ we have:

$$f(x) = \int_0^{1/2} x^Q s^Q e^{qx^\gamma (1-s^\gamma-(1-s)^\gamma)} x ds = x^{Q+1} \int_0^{1/2} t^Q e^{qx^\gamma (1-t^\gamma-(1-t)^\gamma)} dt.$$

There is a classical theorem [12, Section 2.4] for integrals of the type

$$F(x) = \int_\alpha^\beta g(t) e^{xh(t)} dt.$$

In the usual notation, $f(t) \sim g(t)$ as $t \to \alpha$ means that $\lim_{t\to\alpha} \frac{f(t)}{g(t)} = 1$. Suppose that:

h is real-valued and continuous at $t = \alpha$;
h' exists and is continuous for $\alpha < t \le \beta$;
$h' < 0$ for $\alpha < t < \alpha + \eta$ with some $\eta > 0$;
$h(t) \le h(\alpha) - \varepsilon$ for some $\varepsilon > 0$ and all $t \in [\alpha + \eta, \beta]$;
$h'(t) \sim -a(t-\alpha)^{\nu-1}$ as $t \to \alpha$, where $\nu > 0$;
$g(t) \sim b(t-\alpha)^{\lambda-1}$ as $t \to \alpha$, where $\lambda > 0$.

Then

$$F(x) \sim \frac{b}{\nu} \Gamma\Big(\frac{\lambda}{\nu}\Big) \Big(\frac{\nu}{ax}\Big)^{\lambda/\nu} e^{xh(\alpha)}$$

as $x \to \infty$.

We can apply this theorem to

$$F(x) = \int_0^{1/2} t^Q e^{x(1-t^\gamma-(1-t)^\gamma)} dt.$$

In this case $g(t) = t^Q$, $h(t) = 1 - t^\gamma - (1-t)^\gamma$, $\alpha = 0$, $\beta = 1/2$. The derivative $h'(t) = -\gamma\big(t^{\gamma-1} - (1-t)^{\gamma-1}\big)$ is negative on $(0, 1/2)$ since $\gamma - 1 < 0$ (so that $t^{\gamma-1}$ decreases) and $t < 1 - t$. It follows that h decreases, so all the conditions hold. We have $b = 1$, $\lambda = Q + 1$, $a = \nu = \gamma$. Thus,

$$F(x) \sim \frac{1}{\gamma} \Gamma\left(\frac{Q+1}{\gamma}\right) \left(\frac{\gamma}{\gamma x}\right)^{(Q+1)/\gamma} e^{x\cdot 0} =: C_2 x^{-(Q+1)/\gamma}.$$

If we return to f, then we get

$$f(x) = x^{Q+1} F(qx^\gamma) \sim C_2 x^{Q+1} (qx^\gamma)^{-(Q+1)/\gamma} = C_2 q^{-(Q+1)/\gamma} =: C_3.$$

It follows that there is a constant C_4 such that $f(x) \le C_4$ for all $x > 0$.

Now, collecting all together, we have

$$\frac{u * u(x)}{u(x)} \le 2C_0 + C_1 + C_4 \equiv C',$$

what completes the proof.

1.2.4. Fast-growing weights

If the weight is submultiplicative, as we always assume, then it can grow at most exponentially. But $L^p(G, \omega)$ is in general not an algebra with the weight $\omega(x) = e^{|x|}$. We will show this for $G = \mathbb{R}$. Take nonnegative $f, g \in L^p(\mathbb{R}, e^{|x|})$, then $F = e^{|x|} f$, $G = e^{|x|} g$ are in $L^p(\mathbb{R})$;

$$\begin{aligned}(f * g)(s) &= \int_{-\infty}^{\infty} f(t)g(s-t)dt \geq \int_0^s F(t)e^{-t}G(s-t)e^{-s+t}dt \\ &= e^{-s}\int_0^s F(t)G(s-t)dt, \quad \text{for } s \geq 0.\end{aligned}$$

Let $F_+ = F \cdot I_{[0,+\infty)}$, $G_+ = G \cdot I_{[0,+\infty)}$, where $I_{[0,+\infty)}$ is the characteristic function of the interval $[0, +\infty)$. If we assume that $f * g \in L^p(\mathbb{R}, \omega)$, then from the formula above $F_+ * G_+ \in L^p(\mathbb{R})$. Since for every $F_+, G_+ \in L^p([0, +\infty))$ we have $|e^{-t}F_+|, |e^{-t}G_+| \in L^p(\mathbb{R}, \omega)$, it follows that $L^p([0, +\infty))$ is a convolution algebra if $L^p(\mathbb{R}, \omega)$ is so. But this is not true, if $p > 1$. This fact is well-known and can be demonstrated, for example, as follows. For $p > 1$, choose α such that $-\frac{1}{2p} - \frac{1}{2} < \alpha < -\frac{1}{p}$ and define

$$f(x) = \begin{cases} 0, & \text{if } x < 1 \\ x^\alpha, & \text{if } x \geq 1. \end{cases}$$

Then $f \in L^p([0, +\infty))$, but $f * f \sim Cx^{2\alpha+1}$, $x \to +\infty$, and $f * f \notin L^p([0, +\infty))$.

Nevertheless, by 1.2.1 and 1.2.2, $L^p(\mathbb{R}, \omega_1)$ is an algebra for $\omega_1(x) = (1 + |x|)^D e^{|x|}$.

2. First properties of weighted algebras

2.1. Known inequalities

The conditions (LPAlg) guarantee that $L^p(G, \omega)$ is an algebra, that $L^p(G, \omega)$ is translation-invariant and $\|{}^x f\|_{p,\omega} \leq \omega(x)\|f\|_{p,\omega}$, where ${}^x f(t) = f(x^{-1}t)$, as

$$\|{}^x f\|_{p,\omega}^p = \int |f(x^{-1}t)|^p \omega(t)^p dt = \int |f(y)|^p \omega(xy)^p dy \leq \omega(x)^p \|f\|_{p,\omega}^p.$$

Also under (LPAlg) the following is known:

- $L^1(G, \omega) \subset L^1(G)$, and $\|f\|_1 \leq \|f\|_{1,\omega}$ for all $f \in L^1(G, \omega)$
- $L^p(G, \omega) \subset L^1(G)$, and $\|f\|_1 \leq C\|f\|_{p,\omega}$ for all $f \in L^p(G, \omega)$, with $C = \left(\int \omega^{-q}(x)dx\right)^{1/q}$
- $L^1(G, \omega) * L^p(G, \omega) \subset L^p(G, \omega)$, and

$$\|f * g\|_{p,\omega} \leq \|f\|_{1,\omega}\|g\|_{p,\omega} \quad \text{for all } f \in L^1(G, \omega),\ g \in L^p(G, \omega). \tag{2.1}$$

If $\omega(x) = \omega(x^{-1})$, the usual involution $f \mapsto f^*$ is an isometry on $L^p(G, \omega)$: $\|f^*\|_{p,\omega} = \|f\|_{p,\omega}$ (recall that every group of polynomial growth is unimodular).

2.2. Approximate identity

Under the assumption that ω is continuous, the proof of [28] of the fact that the measurable, bounded functions of compact support are dense in $L^p(G,\omega)$, shows that the same is also true for the set $\mathcal{C}_c(G)$ of continuous functions with compact support. By [28], there exists a net $(f_s)_s$ of measurable, bounded functions with compact support which form a bounded approximate identity in $L^1(G,\omega)$ and an (unbounded) approximate identity in $L^p(G,\omega)$. In fact, in the same way it can be proved that if V_s runs through a basis of compact, symmetric neighbourhoods of the identity e in G, then every family f_s such that $0 \le f_s \le 1$, $\|f_s\|_1 = 1$, $\mathrm{supp} f_s \subset V_s$, is an approximate identity with properties as above. It is easy to see that these functions f_s may be chosen to be continuous and self-adjoint, $f_s = f_s^*$. Moreover, it will be convenient to have $V_s \subset K$, where K is a fixed compact set.

2.3. Dual spaces

Let us recall the following definitions and notations.

Definition 2.1. (i) A representation (T, V) of the group G on a non-zero Banach space V is called topologically irreducible if V admits no non-trivial, closed, T-invariant subspaces.

(ii) For any locally compact group G, $\widehat{G}$ will denote the space of equivalence classes of topologically irreducible unitary representations of G. The space $\widehat{G}$ will be equipped with the Fell topology.

For Banach algebras, we have similar definitions:

Definition 2.2. (i) For any Banach algebra $\mathcal{A}$, a representation (T, V) of $\mathcal{A}$ on a non-zero vector space V is said to be algebraically irreducible, if there are no non-trivial T-invariant subspaces in V.

(ii) The set of all kernels of algebraically irreducible representations of $\mathcal{A}$ is denoted by $\mathrm{Prim}\mathcal{A}$ and is equipped with the hull-kernel topology.

(iii) A representation (T, V) of $\mathcal{A}$ on a non-zero Banach space V is said to be topologically irreducible, if there are no non-trivial, closed, T-invariant subspaces in V.

(iv) If $\mathcal{A}$ is a Banach $*$-algebra, the set of all kernels of topologically irreducible $*$-representations of $\mathcal{A}$ is denoted by $\mathrm{Prim}_*\mathcal{A}$ and is equipped with the hull-kernel topology.

(v) The set of equivalence classes of topologically irreducible $*$-representations of the Banach $*$-algebra $\mathcal{A}$ will be denoted by $\widehat{\mathcal{A}}$.

The previous definitions will in particular be used for the algebras $L^1(G)$, $L^1(G,\omega)$ and $L^p(G,\omega)$.

If the group G is abelian, $\widehat{G}$ coincides with the set of all (unitary) characters of G. In this case, the spaces $\mathrm{Prim}L^1(G)$ and $\mathrm{Prim}_*L^1(G)$ coincide and may be identified with $\widehat{G}$, respectively $\widehat{L^1(G)}$.

Unfortunately, for general locally compact groups the spaces $\widehat{L^1(G)}$, $\mathrm{Prim}L^1(G)$ and $\mathrm{Prim}_*L^1(G)$ may be different. The same remark applies for $\widehat{L^p(G,\omega)}$, $\mathrm{Prim}L^p(G,\omega)$ and $\mathrm{Prim}_*L^p(G,\omega)$. In particular, we will have to distinguish between the weak Wiener property (Section 6) and the Wiener property (Section 7).

For any locally compact group G, $\widehat{G}$ and $\widehat{L^1(G)}$ may be identified via integration. The same is true for $\widehat{G}$ and $\widehat{L^p(G,\omega)}$ with G in $[PG]$, as will be shown in Section 2.4.

2.4. Representations of the weighted algebras

For every non-degenerate $*$-representation T of $L^p(G,\omega)$ in a Hilbert space $\mathcal{H}$, there is a unitary continuous representation V of G such that

$$T(f) = \int_G V(x) f(x) dx \tag{2.2}$$

for all f. This is proved exactly as in [21, Theorem 22.7], though the assumption (ii) of this theorem does not hold. Let us make the following remarks on the proof.

(1) If T is cyclic with the cyclic vector ξ, one defines V as follows: on the dense subspace of vectors of the type $T(f)\xi$, $f \in L^p(G,\omega)$, put $V(x)(T(f)\xi) = T({}^x f)\xi$; it may be easily shown that this is an isometry, so $V(x)$ extends to a unitary operator on $\mathcal{H}$. It is also straightforward that V is a representation.

(2) To get the equality (2.2), we need to prove that every coefficient $x \mapsto \langle V(x)T(f)\xi, \eta\rangle$, where $f \in L^p(G,\omega)$, and $\xi, \eta \in \mathcal{H}$, is measurable (this is [21, 22.3i]). But we have even more: coefficients of V are continuous since by [28] the mapping $x \mapsto {}^x f$ from G to $L^p(G,\omega)$ is continuous for every f. From this, by [21, 22.3] we get some representation $\tilde{T}$ of $L^p(G,\omega)$ defined by

$$\tilde{T}(f) = \int_G V(x) f(x) dx. \tag{2.3}$$

(3) To prove that $\tilde{T} = T$, take a vector $\eta \in \mathcal{H}$ and the cyclic vector $\xi \in \mathcal{H}$. For the linear functional $H(f) = \langle T(f)\xi, \eta\rangle$ on $L^p(G,\omega)$ there is a function $h \in L^q(G,\omega)$ such that $H(f) = \int f(x)h(x)dx$. Then for any $f, g \in L^p(G,\omega)$ we have, with all integrals absolutely converging,

$$\begin{aligned}
\langle \tilde{T}(f)T(g)\xi, \eta\rangle &= \int_G \langle V(x)f(x)T(g)\xi, \eta\rangle dx = \int_G \langle V(x)T(g)\xi, \eta\rangle f(x)dx \\
&= \int_G \langle T({}^x g)\xi, \eta\rangle f(x)dx = \int_G \left(\int_G {}^x g(y)h(y)dy\right) f(x)dx \\
&= \int_G \left(\int_G g(x^{-1}y)h(y)dy\right) f(x)dx = \int_G h(y)(f*g)(y)dy \\
&= \langle T(f*g)\xi, \eta\rangle = \langle T(f)T(g)\xi, \eta\rangle.
\end{aligned}$$

It follows that $\tilde{T} = T$ on the dense subspace of vectors of the type $T(f)\xi$, $f \in L^p(G,\omega)$, and as a consequence on the whole $\mathcal{H}$.

(4) In general, T can be expanded into a direct sum of cyclic representations T_α [21, 21.13]. Every T_α is given by the formula (2.3) with some V_α; then T is equal to the same integral (2.3) with $V = \oplus V_\alpha$.

Further, by [21, 22.6], T and V are irreducible or not simultaneously.

In particular, this gives us the identification $\widehat{L^p(G,\omega)} = \widehat{G}$.

3. Symmetry

3.1

The notion of symmetry plays an important role in the theory of Banach $*$-algebras. It may be defined as follows:

Let $\mathcal{A}$ be a Banach $*$-algebra and let $a \in \mathcal{A}$. We will denote the spectrum of a in $\mathcal{A}$ by $\sigma_{\mathcal{A}}(a)$ and the spectral radius of a in $\mathcal{A}$ by $r_{\mathcal{A}}(a)$. Then the algebra $\mathcal{A}$ is said to be *symmetric* if $\sigma_{\mathcal{A}}(a^*a) \subset [0, +\infty[$ for all $a \in \mathcal{A}$, or, equivalently, if $\sigma_{\mathcal{A}}(a) \subset \mathbb{R}$ for all $a = a^* \in \mathcal{A}$.

For abelian Banach $*$-algebras the symmetry is equivalent to the fact that all the characters of the algebra are unitary.

Any C^*-algebra is symmetric (see [38]). If $G = \mathbb{R}^n$, $\mathbb{Z}^n$ or $\mathbb{T}^n$, then $L^1(G)$ is symmetric. The same is true for any connected, simply connected, nilpotent Lie group. All these results, though known separately for some time, are particular cases of a general theorem by Losert [34]: If $G \in [PG]$, then $L^1(G)$ is symmetric.

As noticed by Leptin [33], symmetric Banach $*$-algebras are very important in representation theory of locally compact groups and in the ideal theory of their group algebras. For representation theory of Banach $*$-algebras, the importance of symmetry is due to the following result: A Banach $*$-algebra $\mathcal{A}$ is symmetric if and only if each algebraically irreducible $\mathcal{A}$-module has an inner product relative to which the module action is a $*$-representation [33]. This implies that if $\mathcal{A}$ is symmetric, then $\mathrm{Prim}\mathcal{A} \subset \mathrm{Prim}_*\mathcal{A}$, whereas for general Banach $*$-algebras, $\mathrm{Prim}\mathcal{A}$ and $\mathrm{Prim}_*\mathcal{A}$ may not be compared. See [33] for more details. Another important consequence is that for symmetric Banach $*$-algebras, the weak Wiener property implies the Wiener property (see Sections 6 and 7).

Let now (G, ω) satisfy (LPAlg). Let us recall the following definitions for the weight ω (see for instance [16,15]).

Definition 3.1. (a) The weight ω on G is said to satisfy the GRS-condition or GNR-condition (for Gelfand–Raikov–Šhilov or Gelfand–Naimark–Raikov) if

$$\lim_{n\to+\infty} \omega(x^n)^{\frac{1}{n}} = 1, \quad \forall x \in G. \tag{GRS}$$

(b) The weight ω is said to satisfy condition (S) (where (S) stands for symmetry) if, for every generating, relatively compact neighbourhood U of G,

$$\lim_{n\to+\infty} \sup_{x\in U^n} \omega(x)^{\frac{1}{n}} = 1. \tag{S}$$

These conditions are linked to the symmetry of weighted group algebras. Among others, the following results are known:

For $G = \mathbb{Z}$, $l^1(\mathbb{Z}, \omega)$ is symmetric if and only if ω satisfies the GRS-condition [37].

If $G \in [PG]$ and ω satisfies condition (S), then $L^1(G, \omega)$ is symmetric [16].

The final version of results of this type is due to Fendler et al. [15]. They prove:

Theorem 3.2. *Let $G \in [PG]$ and let ω be a weight on G. Then the following are equivalent:*

(i) *ω satisfies the GRS-condition.*
(ii) *ω satisfies condition (S).*
(iii) *$L^1(G,\omega)$ is symmetric.*
(iv) *$\sigma_{L^1(G,\omega)}(f) = \sigma_{L^1(G)}(f),\ \forall f \in L^1(G,\omega)$.*

The last three results are based on a method developed by Ludwig [35]. Previously, using a result of Hulanicki [22], Pytlik [41] had already proved the following:

Theorem 3.3. *If the weight ω satisfies*

$$\omega(xy) \le C(\omega(x) + \omega(y)), \quad \forall x, y \in G \tag{3.1}$$

for some positive constant C and if $\omega^{-1} \in L^p(G)$ for some $0 < p < +\infty$, then $L^1(G,\omega)$ is symmetric.

Pytlik calls a weight satisfying (3.1) a *polynomial weight*. We will call such a weight *polynomial in the sense of Pytlik*. In particular, weights of the form

$$\omega(x) = (1 + |x|)^D,$$

for some positive D, where

$$|x| := \inf\{n \mid x \in U^n\},$$

satisfy $\omega(xy) \le C(\omega(x) + \omega(y))$ for all $x, y \in G$, and hence give symmetric weighted group algebras $L^1(G,\omega)$, as Pytlik already noticed in [40].

By [41] every weight satisfying $\omega(xy) \le C\big(\omega(x) + \omega(y)\big)$ is dominated by a weight of the form $K(1 + |x|)^D$, $K \ge 1$, $D > 0$. It is then easy to check that all these weights satisfy condition (S). So the result of Pytlik is a particular case of the result of Fendler et al.

3.2

Our aim is to study symmetry for weighted L^p-algebras. For the rest of this section we hence assume that (G,ω) satisfies (LPAlg), in order to be sure that $L^p(G,\omega)$ is an algebra. The question studied in this section is whether condition (S) will also imply the symmetry of $L^p(G,\omega)$. We need some preliminary result.

Lemma 3.4. *The weight ω satisfies condition (S) if and only if $\omega(x) = \mathcal{O}(e^{\varepsilon|x|})$ for all $\varepsilon > 0$, where $|x| = \inf\{n \mid x \in U^n\}$.*

Proof. Let us assume that $\omega(x) \le C(\varepsilon)e^{\varepsilon|x|}$ for some constant $C(\varepsilon)$, $\varepsilon > 0$. Then

$$x \in U^k \Rightarrow |x| \le k$$
$$\Rightarrow \omega(x) \le C(\varepsilon)e^{\varepsilon k}$$

and

$$\sup_{x \in U^k} \omega(x)^{\frac{1}{k}} \le C(\varepsilon)^{\frac{1}{k}} e^{\varepsilon}.$$

So

$$1 \leq \lim_{k\to+\infty} \sup_{x\in U^k} \omega(x)^{\frac{1}{k}} \leq e^{\varepsilon}.$$

As this has to be true for all $\varepsilon > 0$, $\lim_{k\to+\infty} \sup_{x\in U^k} \omega(x)^{\frac{1}{k}} = 1$ and ω satisfies (S).

Conversely, let us assume that there exists $\varepsilon > 0$ such that $\omega(x)$ is not $\mathcal{O}(e^{\varepsilon|x|})$. So, for every $k \in \mathbb{N}$, there exists $x_k \in G$ such that $\omega(x_k) > ke^{\varepsilon|x_k|} > k$. As ω is bounded on each U^n, this implies that "$x_k \to \infty$", which means the following: Let $n(k) := |x_k|$. Then the sequence $(n(k))_k$ admits a subsequence $(\tilde{n}(r))$ such that

$$\begin{aligned}
&\tilde{n}(r) \to +\infty, \quad \text{if } r \to +\infty\\
&x_r \in U^{\tilde{n}(r)} \setminus U^{\tilde{n}(r)-1}\\
&\omega(x_r) > re^{\varepsilon\tilde{n}(r)}\\
&\sup_{x\in U^{\tilde{n}(r)}} \omega(x)^{\frac{1}{\tilde{n}(r)}} \geq \omega(x_r)^{\frac{1}{\tilde{n}(r)}} > r^{\frac{1}{\tilde{n}(r)}} e^{\varepsilon} \geq e^{\varepsilon}.
\end{aligned}$$

Hence

$$\overline{\lim_{r\to+\infty}} \sup_{x\in U^{\tilde{n}(r)}} \omega(x)^{\frac{1}{\tilde{n}(r)}} \geq e^{\varepsilon} > 1.$$

Thus ω does not satisfy condition (S). □

We may now prove the symmetry result for abelian groups:

Theorem 3.5. *Let G be an abelian group such that (G, ω) satisfies (LPAlg). Let us assume that ω satisfies condition (S). Then $L^p(G, \omega)$ is a symmetric Banach $*$-algebra.*

Proof. According to [27], every character χ of $L^p(G, \omega)$ is of the form

$$\chi(f) = \int_G f(x)\sigma(x)dx = \int_G \Big(f(x)\omega(x)\Big)\Big(\sigma(x)\omega^{-1}(x)\Big)dx, \quad \forall f \in L^p(G, \omega),$$

where σ is a (possibly unbounded) character of the group G. As $f\omega \in L^p(G)$ is arbitrary, $|\sigma(x)|\omega(x)^{-1} \in L^q(G)$, with $\frac{1}{p} + \frac{1}{q} = 1$, and, as $\omega(x) \leq C(\varepsilon)e^{\varepsilon|x|}$ for all $\varepsilon > 0$,

$$|\sigma(x)|e^{-\varepsilon|x|} \leq |\sigma(x)|C(\varepsilon)\omega(x)^{-1} \in L^q(G).$$

Let us assume that σ is not unitary. Then there exists $x_0 \in G$ and $\delta > 0$ such that $|\sigma(x_0)| > 1 + 2\delta$. By continuity, there is a non-empty open subset V of G such that $|\sigma(x)| > 1 + \delta$, for all $x \in V$. Hence, for $x \in V^n$, $x = x_1 \cdot x_2 \cdots x_n$ with $x_j \in V$ for all j, and $|\sigma(x)| = \prod_{j=1}^n |\sigma(x_j)| \geq (1 + \delta)^n$. On the other hand, if $V \subset U^k$ where U is a (relatively compact) generating neighbourhood of the identity then $V^n \subset U^{kn}$, and so

$|x| \leq kn$ for $x \in V^n$. This implies that, for all n (and for all $\varepsilon > 0$),

$$+\infty > \int_G \Big(|\sigma(x)|e^{-\varepsilon|x|}\Big)^q dx \tag{3.2}$$

$$\geq \int_{V^n} |\sigma(x)|^q e^{-\varepsilon q|x|} dx \tag{3.3}$$

$$\geq \big(1+\delta\big)^{qn} \int_{V^n} e^{-\varepsilon qkn} dx \tag{3.4}$$

$$\geq \Big(\big(1+\delta\big)\cdot e^{-\varepsilon k}\Big)^{qn} |V|. \tag{3.5}$$

But we can choose ε so that $\big(1+\delta\big)\cdot e^{-\varepsilon k} > 1$, then (3.5) tends to $+\infty$ with n. This is a contradiction which shows that σ, and hence χ are unitary. So $L^p(G,\omega)$ is symmetric. □

Example. $L^2(\mathbb{R}, e^{\sqrt{|x|}})$ is a symmetric Banach $*$-algebra (Section 1.2.3).

3.3

Before studying the non-abelian case, let us first recall the generalized Minkowski inequality: Let X, Y be measure spaces and let F be a measurable function on $X \times Y$. Then, for all $p \geq 1$,

$$\left(\int_X \left(\int_Y |F(x,y)|dy\right)^p dx\right)^{\frac{1}{p}} \leq \int_Y \left(\int_X |F(x,y)|^p dx\right)^{\frac{1}{p}} dy.$$

We need the following relation:

Lemma 3.6. *Let us assume that the weight ω satisfies (LPAlg) and is polynomial in the sense of Pytlik. Then*

$$\|f * g\|_{p,\omega} \leq C\Big(\|f\|_{p,\omega}\, \|g\|_1 + \|g\|_{p,\omega}\, \|f\|_1\Big), \quad \forall f, g \in L^p(G,\omega) \subset L^1(G).$$

Proof.

$$\begin{aligned}
\|f * g\|_{p,\omega} &= \left(\int_G \left|\int_G f(y)g(y^{-1}x)dy\right|^p \omega(x)^p dx\right)^{\frac{1}{p}} \\
&\leq C\left(\int_G \left|\int_G f(y)g(y^{-1}x)\big(\omega(y)+\omega(y^{-1}x)\big)dy\right|^p dx\right)^{\frac{1}{p}} \\
&\leq C\left(\int_G \left|\int_G f(y)g(y^{-1}x)\omega(y)dy\right|^p dx\right)^{\frac{1}{p}} \\
&\quad + C\left(\int_G \left|\int_G f(y)g(y^{-1}x)\omega(y^{-1}x)dy\right|^p dx\right)^{\frac{1}{p}} \\
&= I + II
\end{aligned}$$

by the triangle inequality for $\|\cdot\|_p$. Then the generalized Minkowski inequality implies that

$$\begin{aligned} I &\le C\left(\int_G\left(\int_G |f(y)g(y^{-1}x)|\omega(y)dy\right)^p dx\right)^{\frac{1}{p}} \\ &\le C\left(\int_G\left(\int_G |f(xu^{-1})g(u)|\omega(xu^{-1})du\right)^p dx\right)^{\frac{1}{p}} \quad (y = xu^{-1}) \\ &\le C\int_G\left(\int_G |f(xu^{-1})|^p|g(u)|^p\omega(xu^{-1})^p dx\right)^{\frac{1}{p}} du \\ &= C\|f\|_{p,\omega}\,\|g\|_1 \end{aligned}$$

and

$$\begin{aligned} II &\le C\left(\int_G\left(\int_G |f(y)g(y^{-1}x)|\omega(y^{-1}x)dy\right)^p dx\right)^{\frac{1}{p}} \\ &\le C\int_G\left(\int_G |f(y)|^p|g(y^{-1}x)|^p\omega(y^{-1}x)^p dx\right)^{\frac{1}{p}} dy \\ &= C\|g\|_{p,\omega}\,\|f\|_1. \quad \square \end{aligned}$$

We may now use the methods of Pytlik [40,41] to show the symmetry of the algebra $L^p(G,\omega)$ for polynomial weights. This is done via the following lemmas.

Lemma 3.7. *Let (G,ω) satisfy (LPAlg). Then, for any $f \in L^p(G,\omega) \subset L^1(G)$, $r_1(f) \le r_{p,\omega}(f)$, where $r_1(f)$ denotes the spectral radius of f in $L^1(G)$ and $r_{p,\omega}(f)$ denotes the spectral radius of f in $L^p(G,\omega)$.*

Proof. From

$$\|f\|_1 = \int_G |f(x)|\omega(x)\omega^{-1}(x)dx \le \left(\int_G \omega^{-q}(x)dx\right)^{\frac{1}{q}} \|f\|_{p,\omega} = C\|f\|_{p,\omega}$$

we deduce

$$\|f^{*n}\|_1^{\frac{1}{n}} \le C^{\frac{1}{n}}\|f^{*n}\|_{p,\omega}^{\frac{1}{n}},$$

where $f^{*n} = f * f * \cdots * f$ (n factors). Hence, for $n \to +\infty$,

$$r_1(f) \le r_{p,\omega}(f). \quad \square$$

Lemma 3.8. *Let (G,ω) satisfy (LPAlg). Let us assume that ω is polynomial in the sense of Pytlik. Then*

$$r_1(f) = r_{p,\omega}(f), \quad \forall f \in L^p(G,\omega).$$

Proof. By the methods of Pytlik [41], Lemma 3.6 gives

$$\|f * f\|_{p,\omega} \leq 2C\|f\|_{p,\omega}\,\|f\|_1$$

and, by induction,

$$\|f^{*2^n}\|_{p,\omega} \leq (2C)^n\|f\|_{p,\omega}\,\|f\|_1^{2^n-1}.$$

So,

$$\begin{aligned} r_{p,\omega}(f) &= \lim_{n\to+\infty}\|f^{*2^n}\|_{p,\omega}^{2^{-n}} \\ &\leq \lim_{n\to+\infty}(2C)^{n\cdot 2^{-n}}\|f\|_{p,\omega}^{2^{-n}}\,\|f\|_1^{1-2^{-n}} \\ &= \|f\|_1. \end{aligned}$$

Finally,

$$r_{p,\omega}(f) = r_{p,\omega}(f^{*n})^{\frac{1}{n}} \leq \|f^{*n}\|_1^{\frac{1}{n}}, \quad \forall n,$$

and

$$r_{p,\omega}(f) \leq \lim_{n\to+\infty}\|f^{*n}\|_1^{\frac{1}{n}} = r_1(f). \quad \square$$

We finally get:

Theorem 3.9. *Let (G,ω) satisfy (LPAlg). Let us assume that ω is polynomial in the sense of Pytlik. Then $L^p(G,\omega)$ is a symmetric Banach $*$-algebra.*

Proof. This follows from Lemma 3.1 of [15] applied to $\mathcal{A} := L^p(G,\omega)$ and $\mathcal{B} := L^1(G)$, and from the result of Losert [34] about the symmetry of $L^1(G)$. As a matter of fact, these results imply that

$$\sigma_{L^1(G)}(f) = \sigma_{L^p(G,\omega)}(f), \quad \forall f = f^* \in L^p(G,\omega),$$

and, as $L^1(G)$ is symmetric, $\sigma_{L^1(G)}(f) \subset \mathbb{R}$. $\square$

Example. We know that for $D > 0$ sufficiently large, $\omega(x) = (1+|x|)^D$, gives rise to an L^p-algebra (1.2.1). Hence this algebra is symmetric.

This leads to the following:

Open question: Let (G,ω) satisfy (LPAlg) for some $1 < p < +\infty$. Does condition (S), or equivalently (GRS), imply that the weighted algebra $L^p(G,\omega)$ is symmetric?

As proved in this section, the answer is yes if the group is abelian, or, for non-abelian groups, if the weight ω is polynomial in the sense of Pytlik. The general case is still open.

4. Functional calculus

4.1

Let (G,ω) satisfy (LPAlg). The aim of the following section is the construction of functional calculus for all continuous functions f with compact support such that $f = f^*$.

Functional calculus constructed in this section will be the main tool to study the regularity (Section 5), the weak Wiener property (Section 6) and the Wiener property (Section 7) of our algebras. Let us mention that the idea of functional calculus goes back to Dixmier [9]. We will follow a further development of Dixmier's method found in [23,11] and use their results. To use this method, we have to bound

$$u(nf) := \sum_{k=1}^{\infty} \frac{i^k}{k!} n^k f^{*k}$$

in $L^p(G, \omega)$ and show that there are "enough" functions $\varphi : \mathbb{R} \to \mathbb{R}$, periodic with period 2π, with $\varphi(0) = 0$, such that

$$\varphi\{f\} := \sum_{n\in\mathbb{Z}} u(nf)\hat{\varphi}(n)$$

converges in $L^p(G, \omega)$. For more details on functional calculus, see among others [9,11].

Let us first recall that, for all continuous functions g with compact support, $\|g\|_1 \leq \|g\|_{1,\omega}$ and $\|g\|_1 \leq C\|g\|_{p,\omega}$ for some positive constant C. Moreover,

$$\|u(nf)\| \leq \sum_{k=1}^{+\infty} \frac{1}{k!} n^k \|f\|^k \leq e^{n\|f\|}$$

in any Banach algebra-norm $\|\cdot\|$. So for any continuous function f with compact support such that $f = f^*$, the series defining $u(nf)$ converges in $L^1(G)$, $L^1(G, \omega)$ and $L^p(G, \omega)$ to the same element, i.e. the notation $u(nf)$ represents a function belonging to $L^1(G, \omega) \cap L^p(G, \omega) \subset L^1(G)$. We will now deduce a bound for $\|u(nf)\|_{p,\omega}$ from the bound for $\|u(nf)\|_{1,\omega}$ which was established in [11]. Let us recall the following notations and facts from [11]: There exists a constant $C > 1$ such that

$$\omega(x) \leq e^{C|x|}, \quad \forall x \in G,$$

where $|x| = \inf\{n \mid x \in U^n\}$ for an arbitrary (relatively compact) generating neighbourhood U. We set

$$\begin{aligned}
s(n) &:= \sup_{x\in U^n} \omega(x), \qquad \forall n \in \mathbb{N}^* \\
s(0) &:= 1 \\
\omega_1(x) &:= s(|x|) \\
\omega_2(x) &:= e^{C|x|} \\
s_2(n) &:= \sup_{x\in U^n} \omega_2(x) = e^{Cn}.
\end{aligned}$$

We also consider an arbitrary increasing function $r : \mathbb{N} \to \mathbb{N}$, which will be specified later. It is shown in [11] that there exist positive constants C_1, C_2 such that

$$\|u(nf)\|_{1,\omega} \leq C_1(1 + |n|)(1 + |n|r(|n|))^{\frac{Q}{2}} s(|n|r(|n|))e^{C_2\left(\frac{|n|}{s_2(r(|n|))}\right)}, \quad \forall n \in \mathbb{Z}, \tag{4.1}$$

where Q denotes the power appearing in the polynomial growth condition of the group G, i.e. $|U^n| \leq Kn^Q$, for all $n \in \mathbb{N}^*$, for some positive constant K.

From the formal representation $u(f) = e^{if} - 1$ we get the following identity valid also in the non-unital case:

$$u(nf) = e^{inf} - 1 = e^{i(n-1)f} * e^{if} - 1 = (u((n-1)f) + 1) * (u(f) + 1) - 1 = \\ = u((n-1)f) * u(f) + u((n-1)f) + u(f).$$

By induction it follows that for all $n \in \mathbb{N}^*$

$$u(nf) = nu(f) + \sum_{k=1}^{n-1} u(kf) * u(f).$$

From (2.1), we get an estimate

$$\|u(nf)\|_{p,\omega} \leq n\|u(f)\|_{p,\omega} + \sum_{k=1}^{n-1} \|u(kf)\|_{1,\omega}\|u(f)\|_{p,\omega}.$$

Using the bound for $\|u(kf)\|_{1,\omega}$ obtained in [11] and recalled in (4.1), we get

$$\begin{aligned}\|u(nf)\|_{p,\omega} &\leq Kn + K_1 \sum_{k=1}^{n-1} (1+k)(1+kr(k))^{\frac{Q}{2}} s\big(kr(k)\big) e^{C_2\left(\frac{k}{s_2(r(k))}\right)} \\ &\leq Kn + K_1(n+1)(1+nr(n))^{\frac{Q}{2}} s\big(nr(n)\big) \sum_{k=1}^{n-1} e^{C_2\left(\frac{k}{s_2(r(k))}\right)},\end{aligned}$$

for some constants K, K_1, as the functions s and r are increasing. (Here $1+k$ could be bounded by n as well, but we choose $n+1$ to comply with assumptions of [11]). As in [11], we put $r(n) := \ln(\ln n) + 1$, for $n \geq e^e$. Hence, for $k \geq \max(e^e, e^C)$,

$$s_2(r(k)) = e^{Cr(k)} \geq e^{C\ln(\ln k)} = (\ln k)^C$$

and

$$e^{C_2\left(\frac{k}{s_2(r(k))}\right)} \leq e^{C_2\left(\frac{k}{(\ln k)^C}\right)} \leq e^{C_2\left(\frac{n}{(\ln n)^C}\right)},$$

as the function $f(x) = \frac{x}{(\ln x)^C}$ is increasing for $x \geq e^C$. This allows to estimate the sum over k by $ne^{\frac{C_2 n}{(\ln n)^C}}$. Moreover, as in [11], for $n \geq e^e$,

$$\begin{aligned}&\ln(\ln n) \leq r(n) \leq 2\ln(\ln n) \leq 2n \\ &s\big(nr(n)\big) \leq s(n)^{r(n)} \leq s(n)^{2\ln(\ln n)}.\end{aligned}$$

Finally, noticing that $nf = (-n)(-f)$, we may compute $\|u(nf)\|_{p,\omega}$ even for negative n (by replacing the constants depending on f by the sup of the corresponding constants for f and $-f$). So, there exist positive constants A_1, A_2 (depending on f and ω) such that

$$\|u(nf)\|_{p,\omega} \leq A_1(1+|n|)^2(1+n^2)^{\frac{Q}{2}} s(|n|)^{2\ln(\ln|n|)} e^{A_2\left(\frac{|n|}{(\ln|n|)^C}\right)}$$

for all $|n| \geq \max(e^e, e^C)$. We thus obtain a similar bound as for $\|u(nf)\|_{1,\omega}$, except that the factor $(1+|n|)$ has been replaced by $(1+|n|)^2$. Of course the constants are slightly different too. They depend on f and ω. We may conclude exactly as in [11].

4.2

Let us recall the non-abelian Beurling–Domar condition (BDna), introduced in [11], given by

$$\sum_{n\in\mathbb{N},n\geq e^e} \frac{\big(\ln(\ln n)\big)\ln\big(s(n)\big)}{1+n^2} < +\infty.$$

It is independent of the choice of the generating neighbourhood U used to compute $s(n)$. The (BDna) condition is inspired by the works of Beurling [4,5], Domar [10] and Vretblad [47]. For the sake of completeness, let us repeat some comments which were already made in [11]: In [10], Domar proves that for abelian groups G, the algebra $L^1(G,\omega)$ is regular (see Section 5) if and only if the weight ω satisfies

$$\sum_{n=1}^{\infty} \frac{\ln\omega(x^n)}{n^2} < +\infty, \quad \forall x\in G.$$

In case this is true, $L^1(G,\omega)$ also has the Wiener property (see Section 7). For $G=\mathbb{R}$, Vretblad [47] proves a partial converse of the Wiener property: For a large class of weights ω on $\mathbb{R}$ he shows that if

$$\int_{\mathbb{R}} \frac{\ln\omega(x)}{1+x^2}dx = \infty,$$

then $L^1(\mathbb{R},\omega)$ does not have the Wiener property. But as the group $\mathbb{R}$ is written additively instead of multiplicatively, this is equivalent to Domar's condition. The aim is hence to have a similar condition for our weights in the non-abelian case. The use of $s(n)$ instead of $\omega(x^n)$ in (BDna) does not make a big difference: If $G=\mathbb{R}$, ω increasing on $\mathbb{R}_+$ and $U=[-1,1]$, then $s(n)=\omega(n\cdot 1)$ (in the additive notation). So, roughly speaking, (BDna) differs from the Domar–Vretblad conditions only by the presence of the factor $\ln(\ln n)$. We were not able to eliminate this technical factor. But for rapidly growing weights, the presence of $\ln(\ln n)$ does not affect the convergence very much. So, even in the non-abelian case, our condition (BDna) is very close to the best possible condition which Domar and Vretblad had for the group $\mathbb{R}$. Let us also mention that the condition

$$\int_{\mathbb{R}} \frac{\ln\omega(x)}{1+x^2}dx < +\infty$$

is usually called the non-quasianalytic case.

Let us come back to the weighted groups (G,ω) which satisfy (LPAlg). We then have the following result:

Theorem 4.1. *Let (G,ω) satisfy (LPAlg). Let us assume that moreover the weight ω satisfies the (BDna) condition. Let $f=f^*$ be a continuous function with compact support. Then, given a,b,ε such that $0<a<a+\varepsilon<b-\varepsilon<b<2\pi$, there exists a function $\psi:\mathbb{R}\to\mathbb{R}$, continuous, periodic of period 2π such that $\mathrm{supp}\psi\cap[0,2\pi]\subset[a,b]$, $\psi\equiv 1$ on $[a+\varepsilon,b-\varepsilon]$ and*

$$\sum_{n\in\mathbb{Z}} \|u(nf)\|_{p,\omega}|\hat{\psi}(n)| < +\infty.$$

Hence this defines a function

$$\psi\{f\} := \sum_{n\in\mathbb{Z}} \hat{\psi}(n)u(nf) \in L^p(G,\omega)\cap L^1(G,\omega)$$

and the properties of functional calculus are satisfied, i.e.

$$\chi(\psi\{f\}) = \psi(\chi(f))$$

for every character χ of the abelian Banach $*$-subalgebra of $L^p(G,\omega)$ generated by f,

$$\pi(\psi\{f\}) = \psi(\pi(f)), \quad \forall \pi \in \widehat{L^p(G,\omega)} \equiv \widehat{G},$$
$$(\varphi\psi)\{f\} = \varphi\{f\} * \psi\{f\},$$

if the functions φ and $\varphi\psi$ still have the correct properties to allow functional calculus.

Proof. See [11], pages 337 to 345. Here we use again the argument that if a series converges in $L^1(G,\omega)$ and $L^p(G,\omega)$, then it also converges in $L^1(G)$ and the limit is the same in the three spaces. □

4.3

Examples. (a) If $G \in [PG]$ and $\omega(x) = K(1+|x|)^D$ for $K \geq 1$ and $D > 0$ large enough, then (G,ω) satisfies (LPAlg) (1.2.1). It is easy to check, that ω also verifies (BDna) and so functional calculus exists.

(b) Let (G,ω) satisfy (LPAlg) and let us assume that ω is polynomial in the sense of Pytlik. By [41], such a weight is bounded by a weight of the form $K(1+|x|)^D$ and hence (BDna) is verified. Functional calculus exists.

(c) If $G \in [PG]$, then

$$\omega(x) := e^{C|x|^\gamma}, \quad 0 < \gamma < 1,$$

is such that (G,ω) satisfies (LPAlg) (1.2.3) and the weight ω verifies (BDna) [11]. Functional calculus exists.

Remarks. (a) Condition (BDna) is independent of the choice of the generating neighbourhood U.

(b) Condition (BDna) is only slightly more restrictive than the well known Beurling–Domar condition in the abelian case.

(c) If ω satisfies (BDna), it also verifies condition (S).

See [11] for more details.

Functional calculus is a very useful tool to prove different harmonic analysis properties, as will be shown in the rest of this paper.

5. Regularity

5.1

For abelian Banach algebras, regularity is defined as follows by Šilov [45]: Let $\mathcal{A}$ be an abelian Banach algebra and let $\Delta(\mathcal{A})$ denote the space of characters of $\mathcal{A}$. Then $\mathcal{A}$ is said to be regular if, given any $\varphi \in \Delta(\mathcal{A})$ and any closed set $F \subset \Delta(\mathcal{A})$ not containing φ, there exists $x \in \mathcal{A}$ such that $\hat{x}(\varphi) = \varphi(x) = 1$ and $\hat{x}|_F \equiv 0$, where $\hat{x}$ denotes the Gelfand transform of x.

If G is a locally compact abelian group, then $L^1(G)$ is regular. This is in particular the case for $L^1(\mathbb{R})$ and $L^1(\mathbb{T})$: For instance, by Paley–Wiener theorem, if $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ extends to an entire function of exponential growth, then $\hat{f}$ is compactly supported. By multiplying f by a complex exponential and/or rescaling, we may make the support fit in any given interval. A simple example of such a function f is given by $f(x) = (e^{ix} + e^{-ix} - 2)/x^2$. For the proof of regularity of $L^1(G)$ for a general locally compact abelian group, see for instance [24].

For abelian regular semi-simple Banach algebras one can say a lot about their ideals and about spectral synthesis as initiated by Šhilov [45].

In the non-abelian case $\Delta(\mathcal{A})$ should be replaced by the space $Prim_*\mathcal{A}$.

5.2

As previously, we assume that (G, ω) satisfies (LPAlg). Let $C_c(G)$ denote the set of continuous functions with compact support on G. It is obvious that $C_c(G) \subset L^p(G, \omega) \subset L^1(G)$, that $C_c(G)$ is dense in $L^p(G, \omega)$ and in $L^1(G)$, that for all $\pi \in \widehat{G} \equiv \widehat{L^p(G, \omega)}$,

$$\|\pi(f)\|_{op} \leq \|f\|_1 \leq C\|f\|_{p,\omega}, \quad \forall f \in L^p(G, \omega),$$

resp. $\|\pi(f)\|_{op} \leq \|f\|_1$ for all $f \in L^1(G)$. Moreover, we have seen in the previous section that functional calculus is possible on the self-adjoint elements of $C_c(G)$, provided the weight ω satisfies (BDna). This implies that the arguments of [11], pages 350 and 351) remain valid. In particular, we have the following results:

Theorem 5.1. *Let (G, ω) satisfy (LPAlg). Let us assume that the weight ω verifies (BDna). We then have:*

(i) *The map*

$$\Psi : Prim_* L^1(G) \to Prim_* L^p(G, \omega)$$
$$ker(\pi) \mapsto ker(\pi) \cap L^p(G, \omega)$$

is a homeomorphism.

(ii) *In particular, $Prim_* L^p(G, \omega)$, $Prim_* L^1(G, \omega)$ and $Prim_* L^1(G)$ are homeomorphic.*

(iii) *Given any $\rho \in \widehat{G}$ and any open neighbourhood N of ρ in $\widehat{G}$, resp. any open neighbourhood N_1 of $ker(\rho) \cap L^p(G, \omega)$ in $Prim_* L^p(G, \omega)$, there exists $f \in L^p(G, \omega)$ such that $\rho(f) \neq 0$ and $\pi(f) = 0$ for all $\pi \in \widehat{G} \setminus N$, resp. for all π such that $ker(\pi) \cap L^p(G, \omega) \in Prim_* L^p(G, \omega) \setminus N_1$.*

Proof. See [11]. Part of the argument relies heavily on functional calculus and on the $*$-regularity of groups with polynomial growth. □

Point (iii) of the previous theorem, which is often called Domar's property, corresponds to the regularity of abelian Banach algebras.

6. Weak Wiener property

Let us recall the following definition:

Definition 6.1. Let $\mathcal{A}$ be a Banach algebra.
The algebra $\mathcal{A}$ is said to have the weak Wiener property, if every proper closed two-sided ideal of $\mathcal{A}$ is contained in the kernel of an algebraically irreducible representation.

The algebra $L^1(\mathbb{R})$ has the weak Wiener property. In this case this means that for every proper closed ideal I of $L^1(\mathbb{R})$, the Fourier transforms of functions in I have a common zero, i.e. there exists $a \in \mathbb{R}$ such that

$$I \subset \left\{ f \in L^1(\mathbb{R}) \mid \hat{f}(a) = \int_{-\infty}^{+\infty} f(x)e^{-iax}dx = 0 \right\}.$$

This result implies in particular Wiener's Tauberian theorem [49].

The previous result remains valid for all locally compact abelian groups, the complex exponentials being replaced by the (unitary) characters of the group and the integral being computed with respect to the Haar measure of the group. See [18] and [44].

Let now (G, ω) satisfy (LPAlg) where G is not necessarily abelian and let $(f_s)_s$ be an approximate identity of $L^p(G, \omega)$ with the properties discussed in 2.2.

In [11] it is shown that, provided ω satisfies (BDna), there exists a periodic function φ of period 2π with $\varphi(1) = 1$, $\varphi \equiv 0$ in a neighbourhood of 0, such that $\varphi\{f_s\}$ is defined in $L^1(G, \omega)$. By our section on functional calculus in $L^p(G, \omega)$, the same $\varphi\{f_s\}$ also converges in $L^p(G, \omega)$, i.e. $\varphi\{f_s\} \in L^1(G, \omega) \cap L^p(G, \omega)$ for all s. Moreover, in [11] it is shown that

$$\|\varphi\{f_s\} * f - f\|_{1,\omega} \to 0$$

for all continuous functions f with compact support in G. For any $f, g \in C_c(G)$, $f * g \in C_c(G) \subset L^1(G, \omega) \cap L^p(G, \omega)$ and

$$\|\varphi\{f_s\} * f * g - f * g\|_{p,\omega} \leq \|\varphi\{f_s\} * f - f\|_{1,\omega}\|g\|_{p,\omega}.$$

So

$$\|\varphi\{f_s\} * f * g - f * g\|_{p,\omega} \to 0. \tag{6.1}$$

This gives the following result:

Lemma 6.2. *Under the assumptions above, let I be a proper closed two-sided ideal of $L^p(G, \omega)$. Then there exists s such that $\varphi\{f_s\} \notin I$.*

Proof. Let us assume that $\varphi\{f_s\} \in I$, for all s. Then $\varphi\{f_s\} * f * g \in I$ for all s and all $f, g \in C_c(G)$. As I is closed, the relation (6.1) shows that $f * g \in I$ for all $f, g \in C_c(G)$. But this implies that $I = L^p(G, \omega)$, by density, which is a contradiction. □

We are now able to prove the weak Wiener property:

Theorem 6.3. *Let (G, ω) be (LPAlg). Let us also assume that the weight ω satisfies (BDna). Then the algebra $L^p(G, \omega)$ has the weak Wiener property.*

Proof. The proof is standard, but we repeat it for the sake of completeness. Let I be a proper, closed, two-sided ideal of $L^p(G,\omega)$. Let s and φ be such that $\varphi\{f_s\} \notin I$. Let ψ be another function such that functional calculus $\psi\{f_s\}$ is possible and such that $\psi \equiv 1$ on the support of φ. Such a ψ exists by Theorem 4.1. Then

$$\psi\{f_s\} * \varphi\{f_s\} = (\psi\varphi)\{f_s\} = \varphi\{f_s\}.$$

Let us consider the algebra $\mathcal{A} := L^p(G,\omega)/I$. We have

$$0 \neq \dot{\widehat{\varphi\{f_s\}}} \in \mathcal{A}$$
$$(\dot{\widehat{\psi\{f_s\}}} - 1) * \dot{\widehat{\varphi\{f_s\}}} = 0 \quad \text{in } \mathcal{A} \oplus \mathbb{C},$$

where the dot denotes the equivalence class in the quotient space $L^p(G,\omega)/I$. Hence, by [7] $\dot{\widehat{\psi\{f_s\}}} - 1$ is not invertible in $\mathcal{A} \oplus \mathbb{C}$, i.e. $1 \in \sigma_{\mathcal{A}}(\dot{\widehat{\psi\{f_s\}}})$. So $\dot{\widehat{\psi\{f_s\}}} \notin \mathrm{rad}(\mathcal{A})$, where $\mathrm{rad}(\mathcal{A})$ denotes the radical of $\mathcal{A}$. This implies that there exists an algebraically irreducible representation $(\tilde{T}, V)$ of $\mathcal{A}$ such that $\tilde{T}(\dot{\widehat{\psi\{f_s\}}}) \neq 0$. We then define the non-trivial algebraically irreducible representation (T, V) of $L^p(G,\omega)$ by $T(f) := \tilde{T}(\dot{f})$. By construction, $I \subset ker(T)$. Hence $L^p(G,\omega)$ is weakly Wiener. □

7. Wiener property

We start with the following definition:

Definition 7.1. Let $\mathcal{A}$ be a Banach $*$-algebra.
The algebra $\mathcal{A}$ is said to have the Wiener property, if every proper closed two-sided ideal in $\mathcal{A}$ is contained in the kernel of a topologically irreducible $*$-representation of $\mathcal{A}$ on a Hilbert space.

For abelian groups, let us recall that algebraically irreducible representations and topologically irreducible unitary representations coincide with the (unitary) characters of the group. Hence weak Wiener property and Wiener property coincide for abelian groups. The abelian examples given in Section 6 remain valid for the Wiener property.

For the non-abelian case, let us mention the result of Losert [34] which says that for every locally compact, compactly generated group with polynomial growth, $L^1(G)$ has the Wiener property.

It is well known that every symmetric Banach $*$-algebra which has the weak Wiener property, also has the Wiener property (see [31] and [32]). This leads us to the following result:

Theorem 7.2. *Let (G,ω) satisfy (LPAlg). Then the algebra $L^p(G,\omega)$ has the Wiener property in the following cases:*

(a) *G is abelian and ω satisfies $(BDna)$.*

(b) *G is non-abelian and ω is polynomial in the sense of Pytlik.*

(c) *G is non-abelian and $\omega(x) = K(1+|x|)^D$ for D large enough, with $|x| = \inf\{n \mid x \in U^n\}$, where U is an arbitrary, relatively compact, generating neighbourhood of the identity.*

If one could prove that Property (S) implies the symmetry of the algebra $L^p(G,\omega)$ (open question), then the property (BDna) would imply the Wiener property.

8. Minimal ideals of a given hull

Let $\mathcal{A}$ be a Banach $*$-algebra. Let I be any closed two-sided ideal in $\mathcal{A}$. We define the *hull* of I by

$$h(I) := \{ker(\pi) \in \text{Prim}_*\mathcal{A} \mid I \subset ker(\pi)\}.$$

Given any closed subset C of $\text{Prim}_*\mathcal{A}$, one may ask to characterize the closed ideals I of $\mathcal{A}$ such that $h(I) = C$. A ready example is $I_{max} := \bigcap C = \bigcap\{ker(\pi) \mid ker(\pi) \in C\}$, which is the maximal ideal with hull C. If there is a unique such ideal I, we say that C is a set of spectral synthesis. Usually this is not the case. But quite often there exists a minimal closed ideal I in $\mathcal{A}$ with hull C.

Recall that in the case of the algebra $L^1(\mathbb{R})$ we may identify $\text{Prim}_*L^1(\mathbb{R})$ with $\widehat{L^1(\mathbb{R})} \equiv \widehat{\mathbb{R}}$, i.e. with the set of characters of $\mathbb{R}$, and hence with $\mathbb{R}$ (via the bijection $a \mapsto \chi_a$, $\chi_a(x) := e^{-iax}$). Hence if I is any closed ideal in $L^1(\mathbb{R})$, the hull $h(I)$ may be identified with the set

$$\{a \in \mathbb{R} \mid \hat{f}(a) = 0,\ \forall f \in I\}.$$

Given any closed subset C of $\mathbb{R}$, the maximal ideal with hull C is equal to the set of all L^1-functions whose Fourier transform is identically zero on C. The minimal (closed) ideal of $L^1(\mathbb{R})$ with hull C is then the closure of the set of all functions f whose Fourier transform $\hat{f}$ has compact support disjoint from C. More generally, a similar result is valid for any regular, semi-simple, commutative Banach algebra $\mathcal{A}$. In that case the Fourier transform is replaced by the use of the Gelfand transform [38].

In the non-abelian case, the situation is more complicated.

Let us now assume that (G, ω) satisfies (LPAlg) and let us investigate the existence of minimal ideals of a given hull in $L^p(G, \omega)$. We also suppose that ω satisfies (BDna). For details of the following we refer to [11]. We will use the following notations:

Let us denote by Φ the set of functions φ from $\mathbb{R}$ to $\mathbb{R}$, periodic of period 2π, with $\varphi(0) = 0$, with $\text{supp}\varphi \cap [0, 2\pi]$ being compact and contained in $]0, 2\pi[$, which operate on the set of continuous, self-adjoint functions with compact support in the algebras $L^1(G, \omega)$ and $L^p(G, \omega)$. The construction of such functions is described in more details in [11]. See also Theorem 4.1.

For any compact subset C of $\text{Prim}_*L^p(G, \omega)$ (endowed with the hull-kernel topology), let us define

$$\tilde{C} := \{\pi \in \widehat{G} \mid ker(\pi) \in C\} \quad \text{and} \quad \|f\|_C := \sup_{\pi \in \tilde{C}} \|\pi(f)\|_{op}$$

$$m(C) := \{\varphi\{f\} \mid f = f^*, f \in C_c(G), \|f\|_1 \le 1, \varphi \in \Phi, \varphi \equiv 0 \text{ on a neighbourhood of } [-\|f\|_C, \|f\|_C]\}.$$

Let $j(C)$ be the closed two-sided ideal of $L^p(G, \omega)$ generated by $m(C)$. As in [11], one may prove:

Lemma 8.1. *The hull of $j(C)$ is C.*

Proof. See [11]. □

Theorem 8.2. *Let (G, ω) satisfy (LPAlg). Let the weight ω satisfy (BDna). We also assume that the algebra $L^p(G, \omega)$ is symmetric. Let C be a closed subset of $Prim_* L^p(G, \omega)$ ($\equiv Prim_* L^1(G, \omega) \equiv Prim_* L^1(G)$). There exists a closed two-sided ideal $j(C)$ of $L^p(G, \omega)$ with $h(j(C)) = C$, which is contained in every two-sided closed ideal I with $h(I) = C$.*

This is in particular the case if G is either abelian and ω satisfies (BDna) or if G is non-abelian and ω is polynomial in the sense of Pytlik.

Proof. See [11]. □

9. A commutative, symmetric algebra having infinite-dimensional irreducible representations

Let T denote any quasi-nilpotent operator on a Banach space V without non-trivial invariant subspaces. The existence of such operators was proved by Read [42], who constructed an operator on l^1 with the stated properties. The quasi-nilpotency means that

$$\|T\| \leq 1, \qquad \|T^n\|^{1/n} \to 0, \quad n \to \infty. \tag{9.1}$$

Such an operator T will be used throughout this section to construct an example of a weighted algebra $L^p(\mathbb{R}, \omega)$ for any $p > 1$ which is commutative, symmetric and has infinite-dimensional topologically irreducible representations. Up to our knowledge, the first application to representation theory, using an operator of the previous type, is due Atzmon [1].

9.1. Definition of the weights and symmetry

We will introduce a family of weights on $\mathbb{R}$ which will depend on $p \geq 1$. For $p = 1$, we take as a weight

$$\omega(x) = \max\{\|e^{xT}\|, \|e^{-xT}\|\}. \tag{9.2}$$

Obviously, ω is submultiplicative. Thus, $L^1(\mathbb{R}, \omega)$ is an algebra. For $p > 1$, we put $\omega_1(x) = \omega(x)(1+|x|)^2$; by Sections 1.2.1 and 1.2.2, this is an L^p-algebra weight (possibly after multiplication by a constant). For the rest of this section $\mathcal{A}$ will denote either $L^1(\mathbb{R}, \omega)$ or $L^p(\mathbb{R}, \omega_1)$, $p > 1$.

Next we prove the following estimate:

Lemma 9.1. *Let ω be defined by (9.2). Then $\omega(x) = \mathcal{O}(\exp(\varepsilon|x|))$, $x \to \infty$, for any $\varepsilon > 0$.*

Proof. Let $\varepsilon > 0$ be given. It is enough to estimate $\omega(x)$ for $x > 0$. By (9.1), there is $N(\varepsilon)$ such that $\|T^n\| < \varepsilon^n$ for all $n \geq N(\varepsilon)$. Separate the series into two parts:

$\omega(x) = \Omega_1(x) + \Omega_2(x)$, where

$$\Omega_1(x) = \sum_{n<N(\varepsilon)} \frac{\|T^n\|x^n}{n!},$$

$$\Omega_2(x) = \sum_{n\geq N(\varepsilon)} \frac{\|T^n\|x^n}{n!}.$$

For Ω_2, we have:

$$\Omega_2(x) \leq \sum_{n\geq N(\varepsilon)} \frac{\varepsilon^n x^n}{n!} \leq \exp(\varepsilon x).$$

For Ω_1, since $\|T^n\| \leq 1$, we have:

$$\Omega_1(x) \leq \sum_{n<N(\varepsilon)} \frac{x^n \varepsilon^n}{n!\varepsilon^n} \leq \varepsilon^{-N(\varepsilon)} \sum_{n<N(\varepsilon)} \frac{x^n\varepsilon^n}{n!} \leq \varepsilon^{-N(\varepsilon)} \exp(\varepsilon x).$$

Now,

$$\omega(x) \leq \exp(\varepsilon x)\left(1 + \varepsilon^{-N(\varepsilon)}\right) = C(\varepsilon)\exp(\varepsilon x),$$

what proves the lemma. $\square$

Corollary 9.2. *The algebra $L^1(\mathbb{R}, \omega)$ and every algebra $L^p(\mathbb{R}, \omega_1)$ with $p > 1$ are symmetric.*

Proof. It is easy to see that $\omega_1(x) = \mathcal{O}(\exp(\varepsilon x))$ for any $\varepsilon > 0$ as well. By Lemma 3.4, ω and ω_1 satisfy condition (S), so by Theorems 3.2 and 3.5 the algebras $L^1(\mathbb{R}, \omega)$ and $L^p(\mathbb{R}, \omega_1)$ are symmetric. $\square$

9.2. An infinite-dimensional irreducible representation

Let $\mathcal{A}$ stand for $L^1(\mathbb{R}, \omega)$ or for $L^p(\mathbb{R}, \omega_1)$ if $p > 1$. Now we can put

$$U(f) := \int_{\mathbb{R}} \exp(xT) f(x)dx$$

for any $f \in \mathcal{A}$. First of all, we will show that this integral converges absolutely. We can estimate

$$\|U(f)\| \leq \int_{\mathbb{R}} \|\exp(Tx)\|\, |f(x)|dx \leq \int_{\mathbb{R}} \omega(x)|f(x)|dx.$$

If $p = 1$, this equals $\|f\|_{1,\omega}$. If $p > 1$,

$$\begin{aligned}\int_{\mathbb{R}} \omega(x)|f(x)|dx &= \int_{\mathbb{R}} \omega_1(x)|f(x)|(1+|x|)^{-2}dx \leq \|\omega_1 f\|_p \|(1+|x|)^{-2}\|_q \\ &= C_q\|f\|_{p,\omega_1}.\end{aligned}$$

In both cases, we see that $\|U(f)\| \leq C\|f\|_{\mathcal{A}}$, so U is continuous. Clearly U is a homomorphism.

It remains now to show that (U, V) is a topologically irreducible representation of $L^1(\mathbb{R}, \omega)$, respectively $L^p(\mathbb{R}, \omega_1)$, $p > 1$. This will follow from the fact that closed invariant subspaces for U are invariant under T.

Suppose that $Z \subset V$ is a non-zero invariant subspace for U. Take $z \in Z$, $z \neq 0$. Let I_ε be the indicator function of $[0, \varepsilon]$ and let $\xi_\varepsilon = \varepsilon^{-1} I_\varepsilon$. Then

$$\begin{aligned} U(\xi_\varepsilon) - \mathbb{I} &= \frac{1}{\varepsilon}\int_0^\varepsilon e^{xT}dx - \frac{1}{\varepsilon}\int_0^\varepsilon \mathbb{I}dx = \frac{1}{\varepsilon}\int_0^\varepsilon \left(\sum_{n=0}^\infty \frac{(xT)^n}{n!} - \mathbb{I}\right)dx \\ &= \frac{1}{\varepsilon}\int_0^\varepsilon \sum_{n=1}^\infty \frac{(xT)^n}{n!}dx = \frac{1}{\varepsilon}\int_0^\varepsilon \left(xT + \sum_{n=2}^\infty \frac{(xT)^n}{n!}\right)dx. \end{aligned}$$

Now, assuming that $0 < \varepsilon < 1$, we have

$$\begin{aligned} \left\| U(\xi_\varepsilon) - \mathbb{I} - \frac{\varepsilon}{2}T \right\| &= \left\| \frac{1}{\varepsilon}\int_0^\varepsilon x\,dx\,T - \frac{\varepsilon}{2}T + \frac{1}{\varepsilon}\int_0^\varepsilon \sum_{n=2}^\infty \frac{(xT)^n}{n!}dx \right\| \\ &\le \frac{1}{\varepsilon}\int_0^\varepsilon \sum_{n=2}^\infty \frac{\|(xT)^n\|}{n!}dx \\ &\le \frac{1}{\varepsilon}\int_0^\varepsilon x^2 \sum_{n=0}^\infty \frac{x^n}{(n+2)!}dx \le \frac{1}{\varepsilon}\int_0^\varepsilon x^2 e^x dx \\ &\le \frac{1}{\varepsilon}\int_0^\varepsilon e x^2 dx = \frac{e}{\varepsilon}\cdot\frac{\varepsilon^3}{3} < \varepsilon^2. \end{aligned}$$

Thus, $\varepsilon^{-1}\big(U(\xi_\varepsilon) - \mathbb{I}\big) - \frac{1}{2}T \to 0$, as $\varepsilon \to 0$, or $T = \lim_{\varepsilon\to 0} 2\varepsilon^{-1}\big(U(\xi_\varepsilon) - \mathbb{I}\big)$. If now Z is an invariant subspace for U, then its closure $\bar{Z}$ is invariant for T, so $\bar{Z}$ is trivial, and we are done.

The representation (U, V) does not admit any non-trivial finite rank operator. In fact, by Ludwig [36], the subspace of V

$$V_0 := \operatorname{span}\{U(f)\xi \mid U(f) \text{ finite rank}, \xi \in V\}$$

is an algebraically irreducible $\mathcal{A}$-module. As $\mathcal{A}$ is abelian, either $V_0 = \{0\}$ (no finite rank operators) or $dim(V)_0 = 1$. But as (U, V) is topologically irreducible, this implies that $V = V_0$ is one dimensional, if $V_0 \neq \{0\}$. This is a contradiction.

Remarks. It is clear that this example can be extended to $\mathbb{R}^n$: replace e^{xT} by $e^{\eta(x)T}$ (at least for the operator T constructed by Read), where η is a non-zero linear form on $\mathbb{R}^n$. One can show also that $\omega(x) > C\exp(x/\ln x)$, and so the algebras which we construct are not regular.

Comments: This example is interesting for several reasons. The algebras $L^1(\mathbb{R}, \omega)$ and $L^p(\mathbb{R}, \omega_1)$, $p > 1$, are abelian, which implies that all the topologically irreducible $*$-representations are one-dimensional, i.e. are characters, by Schur's lemma [38]. Because of the symmetry, all the characters are unitary. Nevertheless these algebras admit infinite-dimensional topologically irreducible representations. This suggests that even for abelian, symmetric Banach $*$-algebras, the description of all topologically irreducible representations on Banach spaces is a highly non-trivial, open problem.

Acknowledgement

Authors were supported by the research grant F1R-MTH-PUL-10NCHA of the University of Luxembourg.

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